2020.M29

2020L003A1EL

Coimisiún na Scrúduithe Stáit
State Examinations Commission

Leaving Certificate Examination 2020

Mathematics

Paper 1

Higher Level

2 hours 30 minutes

300 marks

Examination Number

Day and Month of Birth

For example, 3rd February is entered as 0302

Centre Stamp

Do not write on this page

Instructions

There are **two** sections in this examination paper.

Section A	Concepts and Skills	150 marks	6 questions
Section B	Contexts and Applications	150 marks	3 questions

Answer all nine questions.

Write your Examination Number in the box on the front cover.

Write your answers in blue or black pen. You may use pencil in graphs and diagrams only.

This examination booklet will be scanned and your work will be presented to an examiner on screen. Anything that you write outside of the answer areas may not be seen by the examiner.

Write all answers into this booklet. There is space for extra work at the back of the booklet.
If you need to use it, label any extra work clearly with the question number and part.

The superintendent will give you a copy of the *Formulae and Tables* booklet. You must return it at the end of the examination. You are not allowed to bring your own copy into the examination.

You will lose marks if your solutions do not include relevant supporting work.

You may lose marks if the appropriate units of measurement are not included, where relevant.

You may lose marks if your answers are not given in simplest form, where relevant.

Write the make and model of your calculator(s) here:

Section A Concepts and skills 150 marks

Answer **all six** questions from this section.

Question 1 **(25 marks)**

(a) $f(x) = x^2 + 5x + p$ where $x \in \mathbb{R}$, $-3 \leq p \leq 8$, and $p \in \mathbb{Z}$.

(i) Find the value of p for which $x + 3$ is a factor of $f(x)$.

(ii) Find the value of p for which $f(x)$ has roots which differ by 3.

(iii) Find the two values of p for which the graph of $f(x)$ will not cross the x-axis.

(b) Find the range of values of x for which $|2x + 5| - 1 \leq 0$, where $x \in \mathbb{R}$.

Question 2 **(25 marks)**

(a) Find the two complex numbers z_1 and z_2 that satisfy the following simultaneous equations, where $i^2 = -1$:

$$\begin{aligned} iz_1 \quad &= -4 + 3i \\ 3z_1 - z_2 &= 11 + 17i. \end{aligned}$$

Write your answers in the form $a + bi$ where $a, b \in \mathbb{Z}$.

(b) The complex numbers $3 + 2i$ and $5 - i$ are the first two terms of a **geometric** sequence.

(i) Find r, the common ratio of the sequence.
Write your answer in the form $a + bi$ where $a, b \in \mathbb{Z}$.

(ii) Use de Moivre's Theorem to find T_9, the ninth term of the sequence.
Write your answer in the form $a + bi$, where $a, b \in \mathbb{Z}$.

Question 3 **(25 marks)**

(a) $f(x) = 6x - 5$ and $g(x) = \frac{x+5}{6}$. Investigate if $f(g(x)) = g(f(x))$.

$f(g(x))$	$g(f(x))$

Conclusion:

(b) The real variables y and x are related by $y = 5x^2$.

(i) The equation $y = 5x^2$ can be rewritten in the form $\log_5 y = a + b \log_5 x$.
Find the value of a and the value of b.

$a =$ $b =$

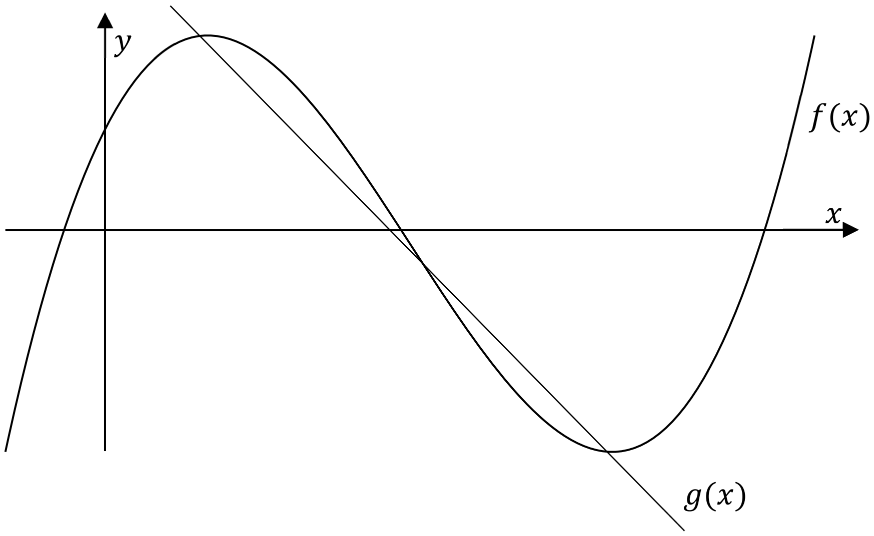
(ii) Hence, or otherwise, find the real values of $\boldsymbol{y}$ for which

$$\log_5 y = 2 + \log_5\left(\frac{126}{25}x - 1\right).$$

Question 4 **(25 marks)**

The diagram below shows two functions $f(x)$ and $g(x)$.

The function $f(x)$ is given by the formula $f(x) = x^3 + kx^2 + 15x + 8$, where $k \in \mathbb{Z}$, and $x \in \mathbb{R}$.



(a) Given that $f'(3) = -12$, show that $k = -9$, where $f'(3)$ is the derivative of $f(x)$ at $x = 3$.

(b) The function $g(x)$ is the line that passes through the two turning points of $f(x) = x^3 - 9x^2 + 15x + 8$, as shown on the previous page.

Find the equation of $g(x)$.

(c) Show that the graph of $g(x)$ contains the point of inflection of $f(x)$.

Question 5 **(25 marks)**

(a) A couple agree to take out a €250 000 mortgage in order to purchase a new home.
The loan is to be paid back monthly over 25 years with the repayments due at the end of each month. The bank charges an annual percentage rate (APR) which is equivalent to a monthly rate of 0·287%.
Using the amortisation formula, or otherwise, find the couples' monthly repayment on the mortgage. Give your answer in euro correct to the nearest cent.

(b) Another couple agree to take out a mortgage of €350 000, at a rate of 0·3% per month, in order to purchase a new home. This loan is also to be paid back monthly over 25 years with the repayments due at the end of each month. The amount of each repayment is €1771.

After exactly 11 years of repayments, the couple receive a financial windfall. They decide to repay the remaining balance on the mortgage.

Write down a **series** (including the first two and last two terms) which shows the total of the present values of all the remaining monthly repayments due over the remaining 14 years of the mortgage (after the last monthly repayment at the end of year 11).

Hence, find how much the couple will need to repay in order to clear their mortgage entirely. Give your answer correct to the nearest cent.

Series:

Repayment:

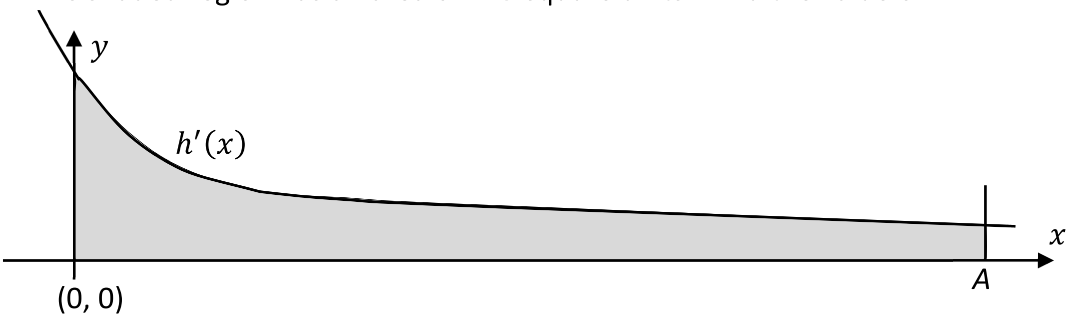
Question 6 **(25 marks)**

(a) Differentiate $(3x-5)(2x+4)$ with respect to x from first principles.

(b) **(i)** $h(x) = \frac{1}{2}\ln(2x + 3) + C$, where C is a constant.

Find $h'(x)$, the derivative of $h(x)$.

(ii) The diagram below shows part of the graph of the function $h'(x)$.
The shaded region in the diagram is between the graph and the x-axis, from $x = 0$ to $x = A$.
This shaded region has an area of $\mathbf{ln3}$ square units. Find the value of A.



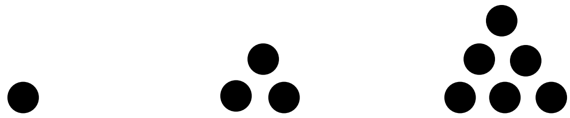
$A =$

Section B   Contexts and Applications   150 marks

Answer **all three** questions from this section.

Question 7 **(50 marks)**

(a) A number of the form $1 + 2 + 3 + \cdots + \ n$ is sometimes called a **triangular number** because it can be represented as an equilateral triangle.
The diagram below shows the first three terms in the sequence of triangular numbers.



$T_1 = 1$ $T_2 = 1 + 2 = 3$ $T_3 = 1 + 2 + 3 = 6$

(i) Complete the table below to list the next five triangular numbers.

Term	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
Triangular Number	1	3	6					

(ii) The n^{th} triangular number can be found directly using the formula

$$T_n = \frac{n(n+1)}{2}.$$

Is 1275 a triangular number? Give a reason for your answer.

Answer:

Reason:

(b) **(i)** The $(n+1)^{\text{th}}$ triangular number can be written as $T_{n+1} = T_n + (n+1)$, where $n \in \mathbb{N}$.

Write the expression $\frac{n(n+1)}{2} + (n+1)$ as a single fraction in its simplest form.

(ii) Prove that the **sum** of any two consecutive triangular numbers will **always** be a square number (a number in the form k^2, where $k \in \mathbb{N}$).

(iii) Two consecutive triangular numbers **sum** to 12 544.

Find the smaller of these two numbers.

This question continues on the next page.

(c) Some numbers are both triangular and square, for example 36.
Leonhard Euler (1778) discovered the following formula for these numbers

$$N_k = \left(\frac{\left(3+2\sqrt{2}\right)^k - \left(3-2\sqrt{2}\right)^k}{4\sqrt{2}}\right)^2$$

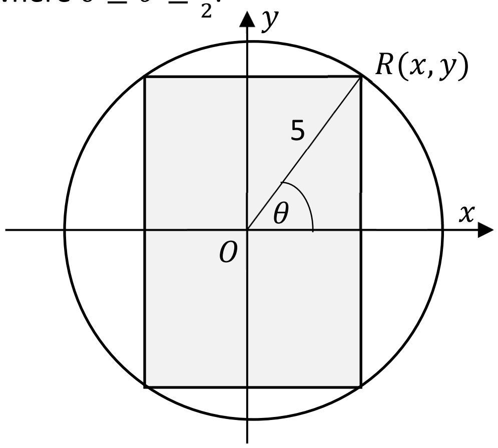
where N_k is the k^{th} number that is both triangular and square.
Use Euler's formula to find N_3, the third number that is both triangular and square.

(d) Prove using **induction** that, for all $n \in \mathbb{N}$, the sum of the first n square numbers can be found using the formula:

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Question 8 **(45 marks)**

A rectangle is inscribed in a circle of radius 5 units and centre $O(0,0)$ as shown below.
Let $R(x,y)$, where $x, y \in \mathbb{R}$, be the vertex of the rectangle in the first quadrant as shown.
Let θ be the angle between $[OR]$ and the positive x-axis, where $0 \leq \theta \leq \frac{\pi}{2}$.



(a) **(i)** The point $R(x,y)$ can be written as $(a\cos\theta, b\sin\theta)$, where $a, b \in \mathbb{R}$.
Find the value of a and the value of b.

$a =$ $b =$

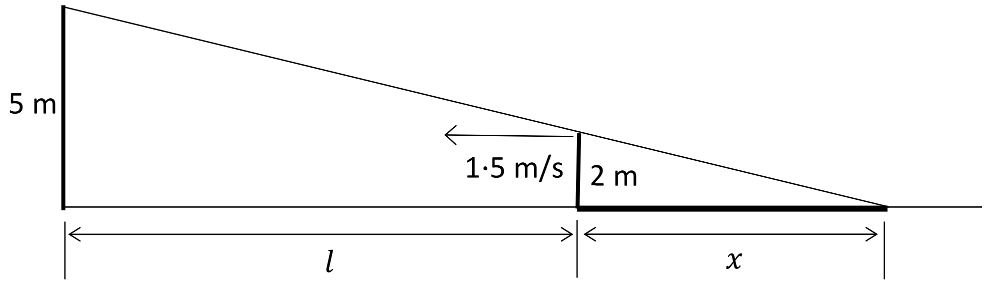
(ii) Show that $A(\theta)$, the area of the rectangle, measured in square units, can be written as $A(\theta) = 50\sin 2\theta$.

(iii) Use calculus to show that the rectangle with maximum area is a square.

(iv) Find this maximum area.

This question continues on the next page.

(b) A person who is 2 m tall is walking towards a streetlight of height 5 m at a speed of 1·5 m/s. Find the rate, in m/s, at which the length of the person's shadow (x), cast by the streetlight, is changing.



Question 9 **(55 marks)**

The number of bacteria in the early stages of a growing colony of bacteria can be approximated using the function:

$$N(t) = 450e^{0\cdot065t}$$

where t is the time, measured in hours, since the colony started to grow, and $N(t)$ is the number of bacteria in the colony at time t.

(a) **(i)** Find the number of bacteria in the colony after 4·5 hours.
Give your answer correct to the nearest whole number.

(ii) Find the time, in **hours**, that it takes the colony to grow to 790 bacteria.
Give your answer correct to 1 decimal place.

This question continues on the next page.

(b) Using the function $N(t) = 450e^{0 \cdot 065t}$, find the average number of bacteria in the colony during the period from $t = 3$ to $t = 12$.
Give your answer correct to the nearest whole number.

(c) Find the rate at which $N(t) = 450e^{0 \cdot 065t}$ is changing when $t = 12$.
Give your answer correct to one decimal place.
Interpret this value in the context of the question.

Rate:

Interpretation:

(d) After k hours, the rate of increase of $N(t)$ is greater than 90 bacteria per hour.
Find the least value of k, where $k \in \mathbb{N}$.

(e) The number of bacteria in the early stages of a **different** colony of bacteria can be approximated using the function:

$$P(t) = 220e^{0 \cdot 17t}$$

where $P(t)$ is the number of bacteria and t is measured in hours.
Assume that both colonies start growing at the same time.
Find the time, to the nearest hour, at which the number of bacteria in both colonies will be equal.

Page for extra work.
Label any extra work clearly with the question number and part.

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Leaving Certificate – Higher Level

Mathematics Paper 1

2 hours 30 minutes